Biometrical Letters
Vol. 46 (2009), No.2, 153-162

Repeated Youden squares with subplot treatments in a proper incomplete block design

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Dedicated to Professor Tadeusz Caliński for his 80th birthday

SUMMARY

This paper deals with a new construction method for row-column designs with split units for two-factor experiments. The whole plot treatments occur in a repeated Youden square. The subplot treatments occur on subplots in a proper incomplete block design. The statistical properties of the final design are examined.

Key words: BIB design, efficiency-balanced block design, general balance, row-column design, Youden square

1. Introduction

In field experiments it is quite common to use a row-column design in order to eliminate real or potential orthogonally disposed heterogeneity of the experimental material. Then a Latin square is the appropriate design. This design possesses many desirable and optimal statistical properties. In the Latin square every treatment occurs once in each row and once in each column. The disadvantage of this design is the large number of experimental units. We can reduce the number of experimental units by using a design in which every treatment occurs once in each row (or each column) only. Then a Youden square is the proper design. This design possesses many desirable statistical properties (see e.g. Cox, 1958). In a Youden square the treatments occur in completely randomized blocks with respect to rows (or columns) while with respect to columns (or rows) they occur in a symmetrical balanced incomplete block design.

2. Construction method

Let us assume that the experimental material is divided into R superblocks. Additionally, each superblock constitutes a row-column design with q rows and a columns. On each unit of the row-column design that is treated as a whole plot, the levels of a factor A $(A_1, A_2, ..., A_a)$ are arranged. These levels will be called whole plot treatments. Additionally, each whole plot is divided into k small plots called subplots, where the levels of the second factor B $(B_1, B_2, ..., B_t)$ are arranged. The levels are then called subplot treatments.

In this paper we will examine the statistical properties of a two-factor design in which each superblock has a Youden square structure with q rows and a columns. Moreover, let us assume that the subdesign of the Youden square with respect to columns is a symmetrical balanced incomplete block design (BIBD) with parameters BIBD($v_A, b_A, r_A, k_A, \lambda_A$). Then the following relationships hold: $k_A = r_A = q$, $b_A = v_A = a$ and $\lambda_A = q(q-1)/(a-1)$.

Let $\mathbf{N}_A$ be the whole plot treatment × column incidence matrix in the Youden square. Then the so-called **C**-matrix for the Youden square subdesign with respect to columns is equal to $\mathbf{C}_A = r_A\mathbf{I} - k_A^{-1}\mathbf{N}_A\mathbf{N}_A'$. Connected with this matrix is the so-called efficiency factor, equal to $\varepsilon_A = a(q-1)/q(a-1)$ with multiplicity $\rho_A = a-1$ (cf. Caliński, Kageyama, 2000).

Let $D_B(t, b, k, \mathbf{N}_B)$ denote any proper block design in which t subplot treatments occur on b blocks of size k according to the incidence matrix $\mathbf{N}_B$, and let $\mathbf{r}_B$ be the vector of treatment replications ($\mathbf{r}_B = \mathbf{N}_B\mathbf{1}$), where **1** stands for the vector of ones and $n_B = bk$.

Let $\mathbf{C}_B = \mathbf{r}_B^\delta - k^{-1}\mathbf{N}_B\mathbf{N}_B'$ be the **C**-matrix of the block design $D_B(t, b, k, \mathbf{N}_B)$ and let ξ_h be an eigenvalue of the matrix $\mathbf{C}_B$ corresponding to an eigenvector $\mathbf{c}_h$ w.r.t. $\mathbf{r}_B^\delta$, i.e. let $\mathbf{C}_B\mathbf{c}_h = \xi_h\mathbf{r}_B^\delta\mathbf{c}_h$, $h = 1, 2, ..., t$. The eigenvectors can be chosen to be $\mathbf{r}_B^\delta$-orthonormal in pairs, i.e. $\mathbf{c}_i'\mathbf{r}_B^\delta\mathbf{c}_i = 1$ and

$\mathbf{c}_i'\mathbf{r}_B^{\delta}\mathbf{c}_h = 0$ for $i \neq h$; i, h = 1, 2,..., t. Since $\mathbf{C}_B\mathbf{1} = \mathbf{0}$, the last eigenvector may be chosen as $\mathbf{c}_t = n_B^{-1/2}\mathbf{1}$.

Let us note that in the incomplete case of the design as is considered here, only a few of the subplot treatments occur on subplots within whole plots. In this work we assume that the contents of whole plots within each superblock are all the same with respect to the subplot treatments, as will be seen in the example in Section 3. Hence we have $R = b$.

The arrangement of whole plot and subplot treatments on the experimental material is based on a proper scheme of randomizations. This scheme includes randomizations of the superblocks, the rows (columns), the columns (rows) and the subplots. As a result of such randomizations and some additional assume-ptions, we can describe the observations by a linear mixed model with random superblock, row and column effects and fixed treatment combination effects (cf. Kachlicka, Mejza, 1996). The applied scheme of randomizations and the structure of the experimental material lead to a linear model of observations, possessing an orthogonal block structure. Then the overall analysis can be split into so-called strata, as in multistratum experiments. In our case we have five strata, i.e. the inter-superblock stratum, inter-column stratum, inter-row stratum, inter-whole plot stratum and finally the inter-subplot stratum (cf. Kachlicka, Mejza, 1996). The present work constitutes an important supplement to the aforementioned paper. In that paper we give a method for analyzing a two-factor experiment carried out on a design in which whole plot treatments occur in repeated row-column designs, while the subplot treatments occur in complete or incomplete randomized block designs, where whole plots are treated as blocks. However, we continue to lack construction methods for the considered class of designs. The papers Kachlicka, Mejza (2004), Kachlicka *et al.* (2004) and Mejza *et al*. (2009) examine the statistical properties of the design in which whole plot treatments occur in a Youden square and subplot treatments occur in a balanced incomplete block design (BIBD) or in a group-divisible block design with two efficiency classes (GD(2)BDs). The BIBD and GD(2)BDs are very useful in biological and agricultural experiments, and hence they are often used to generate new more complex designs with split units (cf. Mejza, Mejza, 1996, Hering, Mejza, 2002, Kachlicka, Mejza, 2003, Ambroży, Mejza, 2004, 2006, Ozawa *et al.*, 2004, Kuriki *et al.*, 2005, Ambroży *et al.*, 2006, Mejza, Ambroży, 2007).

In this paper we examine the statistical properties of a design in which, instead of a BIBD or GD(2)BDs for subplot treatments, any proper incomplete block design is used. In this way, after applying the proper scheme of randomization, the final design also has an orthogonal block structure. The treatment combinations will be considered as treatments with the actually used lexicographical order of combinations $A_s B_h$ (s = 1, 2,…, a; h = 1, 2,…, t) and the usual expression of the treatment effect as the sum of the factor effects and the interaction effects. Let $v = at$ denote the number of treatments.

In the considered design we have 5 main strata in which the statistical analyses may be performed. They are connected with the algebraic properties of stratum information matrices for the treatment combinations, $\mathbf{A}_f$, $f = 1, 2, 3, 4, 5$, with the following forms:

$$\mathbf{A}_1 = (Rqak)^{-1}\left(R\mathbf{N}_0\mathbf{N}_0' - \mathbf{r}\mathbf{r}'\right) = \frac{q}{a}\mathbf{J}_a \otimes \left(b^{-1}\mathbf{N}_B\mathbf{N}_B' - n_B^{-1}\mathbf{r}_B\mathbf{r}_B'\right),$$

$$\mathbf{A}_2 = (qak)^{-1}\left(q\mathbf{N}_1\mathbf{N}_1' - \mathbf{N}_0\mathbf{N}_0'\right) = 0,$$

$$\mathbf{A}_3 = (qak)^{-1}\left(a\mathbf{N}_2\mathbf{N}_2' - \mathbf{N}_0\mathbf{N}_0'\right) = \frac{a-q}{(a-1)k}\left(\mathbf{I}_a - \tfrac{1}{a}\mathbf{J}_a\right) \otimes \mathbf{N}_B\mathbf{N}_B',$$

$$\mathbf{A}_4 = (qak)^{-1}\left(\mathbf{N}_0\mathbf{N}_0' + qa\mathbf{N}_3\mathbf{N}_3' - a\mathbf{N}_2\mathbf{N}_2' - q\mathbf{N}_1\mathbf{N}_1'\right)$$

$$= \frac{a(q-1)}{(a-1)k}\left(\mathbf{I}_a - \tfrac{1}{a}\mathbf{J}_a\right) \otimes \mathbf{N}_B\mathbf{N}_B',$$

$$\mathbf{A}_5 = \mathbf{r}^{\delta} - k^{-1}\mathbf{N}_3\mathbf{N}_3' = q\mathbf{I}_a \otimes \left(\mathbf{r}_B^{\delta}\mathbf{I}_b - \tfrac{1}{k}\mathbf{N}_B\mathbf{N}_B'\right),$$

where

$$\mathbf{r}\mathbf{r}' = q^2\mathbf{J}_a \otimes \mathbf{r}_B\mathbf{r}'_B, \qquad \mathbf{r}^{\delta} = q\mathbf{I}_a \otimes \mathbf{r}_B^{\delta},$$
$$\mathbf{N}_0\mathbf{N}_0' = q^2\mathbf{J}_a \otimes \mathbf{N}_B\mathbf{N}_B', \qquad \mathbf{N}_1\mathbf{N}_1' = q\mathbf{J}_a \otimes \mathbf{N}_B\mathbf{N}_B',$$
$$\mathbf{N}_2\mathbf{N}_2' = \mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{N}_B\mathbf{N}_B', \qquad \mathbf{N}_3\mathbf{N}_3' = q\mathbf{I}_a \otimes \mathbf{N}_B\mathbf{N}_B',$$
$$\mathbf{N}_A\mathbf{N}_A' = (r_A - \lambda_A)\mathbf{I}_a + \lambda_A\mathbf{J}_a = \frac{q(a-q)}{a-1}\mathbf{I}_a + \frac{q(q-1)}{a-1}\mathbf{J}_a.$$

The matrix $\mathbf{J}_t$ denotes the $t \times t$ matrix of ones; $\mathbf{N}_0$, $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$ are the incidence matrices: treatments vs. superblocks, treatments vs. columns, treatments vs. rows and treatments vs. whole plots, respectively; $\mathbf{r}$ denotes the vector of treatment replicates; and $\mathbf{r}^{\delta}$ stands for the diagonal matrix with diagonal elements equal to the numbers of treatment replicates**.**

A desirable statistical property examined here is called general balance (cf. Bailey, 1994). A design is generally balanced if (cf. Mejza, 1992) the information matrices satisfy the conditions:

$$\mathbf{A}_f\mathbf{r}^{-\delta}\mathbf{A}_{f'} = \mathbf{A}_{f'}\mathbf{r}^{-\delta}\mathbf{A}_f, \quad f \neq f', \; f, f' = 1, 2, 3, 4, 5.$$

From the structure of the information matrices $\mathbf{A}_f$ we notice that the considered design is generally balanced.

Let $\mathbf{p}_j$ define contrasts of the form $\mathbf{p}'_j\boldsymbol{\tau}$, j =1, 2,..., *v*-1, in the final design. Let us note that the structures of the information matrices $\mathbf{A}_f$ are based on the Kronecker product of submatrices. Hence the vectors $\mathbf{p}_j$ defining contrasts can be expressed by the Kronecker product of some contrasts corresponding to the eigenvectors of submatrices $\mathbf{C}_A$ and $\mathbf{C}_B$ for the factors A and B respectively. Then $\mathbf{p}'_j\boldsymbol{\tau}$ are called the basic contrasts (cf. Pearce et al., 1974). Next, the eigenvalues of the information matrices $\mathbf{A}_f$ can be identified as stratum efficiency factors of the design with respect to the *j*-th basic contrast in the *f*-th stratum, f = 1, 2, 3, 4, 5, j =1, 2,..., *v*-1 (cf. Houtman, Speed, 1983).

The stratum efficiency factors measure the amount of information that is included in the strata for estimating the treatment contrast. A stratum efficiency factor equal to 1 means that the particular contrast is estimable only in that stratum and with full efficiency. A stratum efficiency factor equal to 0 means that the contrast is not estimable in that stratum (it is confounded). The sum of the stratum efficiency factors for one treatment contrast is equal to 1. In the case of the design under consideration, the stratum efficiency factors, being functions of ε_A and ξ_h, h = 1, 2,..., t-1, are given in Table 1.

Remarks:

- In the first stratum the contrasts among the subplot treatment effects are estimated only with respective efficiency factors of $1-\xi_h$, h = 1, 2,..., t-1.
- In the second stratum no contrast is estimable.
- In the third stratum, the contrasts among the whole plot treatment effects are estimated with the efficiency factor $1-\varepsilon_A$, and the interaction contrasts are estimated with respective efficiency factors of $(1-\varepsilon_A)(1-\xi_h)$, h = 1, 2,..., t-1.
- In the fourth stratum all contrasts among the whole plot treatment effects are estimated with the efficiency factor ε_A, and all interaction contrasts are estimated with respective efficiency factors of $\varepsilon_A(1-\xi_h)$, h = 1, 2,..., t-1.
- In the fifth stratum there are all contrasts among the subplot treatment effects and interaction contrasts which are estimated with respective efficiency factors of ξ_h, where h = 1, 2,..., t-1.

Table 1. Stratum efficiency factors for the considered design

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	a-1	0	0	$1-\varepsilon_A$	ε_A	0
B_h h = 1, 2,..., t-1	1	$1-\xi_h$	0	0	0	ξ_h
$(A\times B)_h$ h = 1, 2,..., t-1	a-1	0	0	$(1-\varepsilon_A)(1-\xi_h)$	$\varepsilon_A(1-\xi_h)$	ξ_h

3. Application

Let us consider a two-factor experiment in which we have four whole plot treatments (a=4), and three subplot treatments (t=3). The experiment was set up in four (R=4) superblocks, each divided into three (q=3) rows and four (a=4) columns. Additionally, each whole plot is divided into three (k=3) subplots. The arrangement of whole plots in a Youden square is the following:

A_2	A_4	A_3	A_1
A_3	A_1	A_4	A_2
A_4	A_2	A_1	A_3

The whole plot treatments are arranged in a BIBD on columns with the incidence matrix :

$$\mathbf{N}_A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The parameters of this subdesign with respect to the whole plot treatments are the following: $k_A = r_A = q = 3$, $b_A = v_A = a = 4$, $\lambda_A = q(q-1)/(a-1) = 2$, $\varepsilon_A = a(q-1)/q(a-1) = 8/9$, $\rho_A = 3$.

The subplot treatments occur in an efficiency-balanced block design with unequal numbers of replicates. The arrangement of the subplot treatments in superblocks can be schematically represented as follows:

B_2	B_1	B_1	B_1
B_3	B_2	B_3	B_2
B_3	B_3	B_3	B_3

In each superblock the subplot treatments are arranged on the subplots within the whole plots according to the incidence matrix

$$\mathbf{N}_B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

Hence we have: $v_B = t = 3$, $R = b_B = 4$, $k = 3$, $\mathbf{r}_B = [3,\ 3, 6]'$, $\xi_h = 8/9$, h = 1, 2, $\xi_3 = 0$. The arrangement of subplot and whole plot treatments in superblocks is given in Table 2.

Table 2. Arrangement of the whole plot and subplot treatments in superblocks (before randomization)

Superblock I

	A_2			A_4			A_3			A_1	
B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3
	A_3			A_1			A_4			A_2	
B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3
	A_4			A_2			A_1			A_3	
B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3	B_2	B_3	B_3

Superblock II

	A_2			A_4			A_3			A_1	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	A_3			A_1			A_4			A_2	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	A_4			A_2			A_1			A_3	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3

Superblock III

	A_2			A_4			A_3			A_1	
B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3
	A_3			A_1			A_4			A_2	
B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3
	A_4			A_2			A_1			A_3	
B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3	B_1	B_3	B_3

Superblock IV

	A_2			A_4			A_3			A_1	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	A_3			A_1			A_4			A_2	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
	A_4			A_2			A_1			A_3	
B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3

The stratum efficiency factors for the above design are given in Table 3.

Table 3. Stratum efficiency factors in the example

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	3	0	0	1/9	8/9	0
B_h $h = 1, 2$	1	1/9	0	0	0	8/9
$A \times B_h$ $h = 1, 2$	3	0	0	1/81	8/81	8/9

4. Final remarks.

In the example incomplete block subdesigns for the whole plot treatments and the subplot treatments are efficiency-balanced with the same efficiency factors. This means that in the final two-factor design, all the basic contrasts among the main effects of both factors A and B are estimated with the same efficiency factors but in different strata. In practice we can use any proper block design for the subplot treatments instead of an efficiency-balanced block design. In particular we can find a design for the subplot treatments satisfying all the experimenter's requirements concerning statistical properties, especially in the area of efficiency.

Acknowledgements

The authors would like to thank to JSPS and PAN (Polish Academy of Sciences) for giving them the opportunity to carry out this joint research in Japan and in Poland. The research was also partially supported by JSPS, Grant-in-Aid for Scientific Research (C)21540142.

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